International Journal of Theoretical Physics, Vol. 31, No. 8, 1992

The Multimomentum Hamiltonian Formalism in Gauge Theory

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Received September 16, 1991

We extend the jet bundle machinery of gauge theory to the multimomentum Hamiltonian formalism. This enables us to manipulate finite-dimensional momentum spaces of fields. In the framework of this formalism, time and spatial coordinates are regarded on the same footing, and a preliminary $(3+1)$ splitting of a world manifold is not required. We get the canonical splitting of a multimomentum Hamiltonian form into a connection part and a Hamiltonian density.

1. INTRODUCTION

In bundle terms, matter fields ϕ are represented by sections of a vector bundle $E \to X$ and their Lagrangian is defined on the 1-jet manifold J^1E of E by the morphism

$$\begin{gathered} L: \quad J^1E \to \overset{n}{\wedge} T^*X \\ L = \mathscr{L}\omega, \qquad \omega = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_n} \end{gathered} \tag{1}$$

(Saunders, 1989; Mangiarotti and Modugno, 1991). The jet manifold is endowed with adapted coordinates,

$$(x^\lambda, y^i, y^i_\lambda) \circ j^1_x\phi = (x^\lambda, \phi^i(x), \partial_\lambda \phi^i(x)) \tag{2}$$

and it plays the role of a finite-dimensional configuration space of fields ϕ.

A finite-dimensional momentum space of fields ϕ is represented by the Legendre manifold

$$\Pi = \overset{n}{\wedge} T^*X \otimes TX \underset{E}{\otimes} V^*E \tag{3}$$

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0020-7748/92/0800-1477$06.50/0

on which the multimomentum Liouville form

$$\theta = -p_i^\lambda \, dy^i \wedge \omega \otimes \partial_\lambda \tag{4}$$

is defined. The manifold (3) is provided with standard coordinates $(x^\lambda, y^i, p_i^\lambda)$. Given the Lagrangian (1), we have the Legendre morphism

$$\begin{gathered} \hat{L}: \quad J^1E \to \Pi \\ (x^\lambda, y^i, p_i^\lambda) \circ \hat{L} = (x^\lambda, y^i, p_i^\lambda = \pi_i^\lambda = \partial_i^\lambda \mathscr{L}) \end{gathered} \tag{5}$$

A multimomentum Hamiltonian form H on Π is defined to be an object of the following kind:

$$\begin{gathered} H: \quad \Pi \to \overset{n}{\wedge} T^*E \\ H = p_i^\lambda \, dy^i \wedge \omega_\lambda - \mathscr{H}(x^\lambda, y^i, p_i^\lambda)\omega, \qquad \omega_\lambda = \partial_\lambda \rfloor \omega \end{gathered} \tag{6}$$

where local functions $\mathscr{H}$ on Π obey the coordinate transformation law

$$\mathscr{H}'(x'^\lambda, y'^i, p_i'^\lambda) = J\left(\frac{\partial y^i}{\partial y'^j} \frac{\partial y'^j}{\partial x^\lambda} p_i^\lambda + \mathscr{H}\right), \qquad J = \det\left(\frac{\partial x^\mu}{\partial x'^\lambda}\right) \tag{7}$$

If $X = \mathbb{R}$, we have the familiar Hamiltonian formalism.

The multimomentum Hamiltonian formalism (Kolář, 1973; Krupka, 1978) has been generalized to degenerate Lagrangian systems (Zakharov, 1991). To apply this formalism to field theory, we use the notion of a general connection on a bundle E:

$$\Gamma: \quad E \to J^1E \tag{8}$$

As a consequence, there is a canonical splitting

$$\mathscr{H} = p_i^\lambda \Gamma_\lambda^i + \tilde{\mathscr{H}}$$

of the multimomentum Hamiltonian form (6), where Γ is some connection (8).

We further assume that all maps are smooth and manifolds are real, Hausdorff, second-countable, finite-dimensional, and connected. Bundles are locally trivial and differentiable. Their structure groups are assumed to be Lie groups.

By $\wedge$, we denote exterior product of cotangent vectors. The interior product (pairing) of tangent vectors with cotangent vectors is denoted by $\rfloor$.

2. BUNDLES

By a bundle, we mean a locally trivial fiber bundle

$$\pi: \quad E \to B$$

whose total space E and base B are manifolds. For the sake of simplicity, we denote a bundle by its total space E.

We use y and x in order to denote points of E and B, respectively.

Given a bundle E and another bundle

$$\pi': \quad E' \to B'$$

a bundle morphism of E to E' is defined to be a pair of manifold morphisms

$$\Phi: \quad E \to E', \qquad \Phi_B: \quad B \to B'$$

such that

$$\pi' \circ \Phi = \Phi_B \circ \pi'$$

One says that Φ is a bundle morphism over Φ_B.

Given a bundle E and a manifold morphism

$$f: \quad B' \to B$$

the pullback of E by f is defined to be the bundle

$$f^*E = \{(y, x') \in E \times B';\ \pi(y) = f(x')\}$$

with the base B' and projection

$$f^*(\pi): \quad (y, x') \to x'$$

In particular, each section e of E yields the pullback section of f^*E:

$$f^*e(x') = (e(f(x')), x')$$

We provide a bundle E with local bundle coordinates

$$(x^\lambda, y^i), \quad 1 \le \lambda \le n = \dim B, \qquad 1 \le i \le l = \dim E - \dim B$$

which are compatible with the bundle fibration of E. In particular, if

$$\Psi = \{U_\kappa, \psi_\kappa: \pi^{-1}(U_\kappa) \to U_\kappa \times F\}$$

is a bundle atlas of E, coordinates y^i on E can be induced by coordinates v^i on a standard fiber F of the bundle E:

$$y^i = v^i \circ \psi_\kappa \tag{9}$$

In field theory, one is usually concerned with bundles associated with a principal bundle.

A group bundle is defined to be a bundle E together with canonical bundle morphisms which make each fiber $E_x = \pi^{-1}(x)$ of E into a Lie group. For instance, a vector bundle E possesses the structure of an additive group bundle.

A general affine bundle is defined to be the triple (E, E', r) of a bundle E, a group bundle E' over B, and a bundle morphism

$$r: \quad E \underset{B}{\times} E' \to E$$

which makes each fiber E_x of E into a general affine space with the associated group E'_x acting freely and transitively on E_x.

In particular, if a group bundle is a vector bundle $\bar{E}$, a general affine bundle is called an affine bundle modeled on the vector bundle $\bar{E}$:

$$r_E: \quad E \underset{B}{\times} \bar{E} \to E$$

$$r_E: \quad (y, \bar{y}) \to y + \bar{y}$$

A principal bundle P with a structure group G is a general affine bundle with respect to the trivial group bundle $B \times G$, where the group G acts on P on the right:

$$r_g: \quad P \to Pg = r(P, g), \qquad g \in G \tag{10}$$

Given a principal bundle

$$\pi_P: \quad P \to B$$

with a structure group G, a total space of a P-associated bundle E with a standard fiber F is defined to be the quotient $(P \times F)/G$ of the product $P \times F$ by identification of elements (p, v) and $(pg, g^{-1}v)$ for all $g \in G$. A global section e of E then is determined by an F-valued equivariant function f_e on P such that

$$e(\pi_P(p)) = [p]_F f_e(p), \qquad p \in P$$

$$f_e(pg) = g^{-1} f_e(p), \qquad g \in G$$

where $[p]_F$ denotes the restriction of the canonical map

$$P \times F \to E$$

to the subset $p \times F$.

Let (E_1, E'_1, r_1) and (E_2, E'_2, r_2) be general affine bundles. An affine bundle morphism $E_1 \to E_2$ is a pair of bundle morphisms

$$\Phi: \quad E_1 \to E_2, \qquad \Phi': \quad E'_1 \to E'_2$$

such that

$$r_2 \circ (\Phi, \Phi') = \Phi \circ r_1$$

For instance, let P be a principal bundle with a structure group G. Every affine (principal) isomorphism of P (over the identity morphism of its base B) is expressed as

$$\begin{aligned} \Phi_P(p) &= pf(p), \qquad p \in P \\ f(pg) &= g^{-1}f(p)g, \qquad g \in G \end{aligned} \tag{11}$$

where f is a G-valued equivariant function on P.

Given a P-associated bundle E with a standard fiber F, every principal isomorphism (11) yields the principal morphism

$$\Phi_E: \quad (P \times F)/G \to (\Phi_P(P) \times F)/G \tag{12}$$

of the bundle E.

Given a principal bundle P and a P-associated bundle E, we say that a bundle atlas

$$\Psi^P = \{U_\kappa, \psi_\kappa^P\}$$

of P and a bundle atlas

$$\Psi = \{U_\kappa, \psi_\kappa\}$$

of E are associated atlases if they are determined by the same family $\{z_\kappa(x), x \in U_\kappa\}$ of local sections of P, that is,

$$\begin{gathered} z_\kappa(\pi_P(p)) = p(\psi_\kappa^P(p))^{-1} = p(\psi_\kappa^P(x))^{-1} 1_G \\ \psi_\kappa(x) = [z_\kappa(x)]_F^{-1}, \qquad \pi_P(p) = x \in U_\kappa \end{gathered}$$

Here, 1_G is the unit element of the group G.

The tangent bundle over a bundle E possesses additional structure, which is the vertical subbundle.

Given the tangent bundle

$$\pi_M: \quad TM \to M$$

and the cotangent bundle T^*M over a manifold M, we denote the induced bundle coordinates on TM and T^*M by $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$, respectively. Here, $\dot{x}^\lambda$ and $\dot{x}_\lambda$ are coordinates on fibers T_xM and T_x^*M with respect to holonomic bases $\{\partial_\lambda\}$ and $\{dx^\lambda\}$. Let

$$f: \quad M \to N$$

be a manifold morphism. This yields the linear bundle morphism over f:

$$\begin{gathered} f_*: \quad TM \to TN \\ f_*: \quad \tau^\mu \partial_\mu \to \tau^\mu \frac{\partial f^\nu}{\partial x^\mu} \partial_\nu \end{gathered}$$

which is called the tangent morphism to f.

Given a bundle E, we have the bundles

$$\pi_E: \quad TE \to E$$

$$\pi_*: \quad TE \to TB$$

The induced bundle coordinates on TE are $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$.

The vertical bundle over a bundle E is defined to be the subbundle

$$VE = \ker \pi_* \subset TE$$

The induced bundle coordinates on VE are $(x^\lambda, y^i, \dot{y}^i)$

We have the following exact sequence of tangent bundles:

$$0 \to VE \to TE \to E \underset{B}{\times} TB \to 0 \tag{13}$$

over E, where

$$E \underset{B}{\times} TB = \pi^*(TB)$$

is the pullback of the tangent bundle TB by π. For instance, a bundle morphism Φ of a bundle E yields the vertical tangent morphism

$$V\Phi = \Phi_*|_{VE}: \quad VE \to VE'$$

of the vertical bundle VE to VE'.

The dual exact sequence of cotangent bundles is

$$0 \to \pi^*(T^*B) \to T^*E \to V^*E \to 0$$

Here, V^*E is the vertical cotangent bundle dual to VE and

$$H^*E = \pi^*(T^*B) = E \underset{B}{\times} T^*B$$

is the horizontal cotangent subbundle of T^*E which consists of covectors whose interior product with vertical tangent vectors is equal to zero. For the sake of simplicity, we denote the horizontal subbundle H^*E by T^*B.

A vector field on E is called a projectable vector field if it is projected to a vector field on B. The coordinate expression of a projectable vector field is

$$u = u^\mu(x)\partial_\mu + u^i(y)\partial_i$$

A projectable vector field on E taking its values in the vertical bundle VE is called a vertical vector field. Its coordinate expression reads

$$u = u^i(y)\partial_i$$

Vertical bundles of most of the bundles relevant for physics possess a simple structure called vertical splitting.

A vertical splitting of a bundle E is made up of a vector bundle $\bar{E}$ and a linear bundle isomorphism over E:

$$\alpha: \quad VE \to E \underset{B}{\times} \bar{E} \tag{14}$$

In particular, a trivial vertical splitting of a bundle E is a vertical splitting with a trivial bundle $\bar{E} = B \times \bar{F}$:

$$\alpha: \quad VE \to E \times \bar{F} \tag{15}$$

Given the vertical splitting (14), the bundle coordinates (x^λ, y^i) on E are called the coordinates adapted to the vertical splitting if the vector fields

$$\mathrm{pr}_2 \circ \alpha \circ \partial_i: \quad E \to VE \to E \underset{B}{\times} \bar{E} \to \bar{E}$$

are constant along fibers of E. In this case, we can write

$$\mathrm{pr}_2 \circ \alpha \circ \partial_i = t_i(x)$$

where the $t_i(x)$ are bases associated with some local splitting ψ of $\bar{E}$. The vertical splitting (14) is called an integrable vertical splitting if there exists a bundle coordinate atlas of E constituted by coordinate charts adapted to the vertical splitting.

For instance, a vector bundle E has a canonical integrable vertical splitting:

$$VE = E \times E \tag{16}$$

An affine bundle E modeled on a vector bundle $\bar{E}$ has a canonical integrable vertical splitting (14).

A principal bundle P with a structure group G has a canonical trivial vertical splitting (15):

$$\alpha: \quad VP \to P \times \mathfrak{g}$$

$$\mathrm{pr}_2 \circ \alpha \circ \partial_m = J_m$$

where $\mathfrak{g}$ is the left Lie algebra of the group G and $\{J_m\}$ is a basis for $\mathfrak{g}$. This splitting takes place because, by definition, elements of the left Lie algebra $\mathfrak{g}$ are left-invariant vector fields on G. Given an atlas $\{z_\kappa\}$ of the bundle P, the canonical bundle coordinates on P adapted to a canonical vertical splitting are (x^λ, p^m)

$$p^m(p) = (a^m \circ \psi_\kappa^P)(p) = a^m(g_p), \qquad p \in \pi_P^{-1}(U_\kappa) \tag{17}$$

where $a^m(g)$ are group parameters and the element $g_p \in G$ is determined by the relation

$$p = z_\kappa(\pi_P(p))g_p$$

In the case of bundles, the familiar machinery of $\mathbb{R}$-valued exterior forms is extended to tangent-valued forms.

A tangent-valued form ϕ on a manifold M is defined to be a section of the bundle

$$\Lambda T^*M \underset{M}{\otimes} TM$$

Its coordinate expression is

$$\phi = \phi^{\mu}_{\lambda_1 \cdots \lambda_r} \partial_\mu \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$

Given a bundle E, one manipulates the following classes of tangent-valued forms on E:

(i) Tangent-valued horizontal forms

$$\phi\colon \quad E \to \Lambda T^*B \underset{E}{\otimes} TE$$

$$\phi = (\phi^{\mu}_{\lambda_1 \cdots \lambda_r}(y)\partial_\mu + \phi^{i}_{\lambda_1 \cdots \lambda_r}(y)\partial_i) \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$

(ii) Projectable horizontal forms projected to tangent-valued forms on B:

$$\phi = (\phi^{\mu}_{\lambda_1 \cdots \lambda_r}(x)\partial_\mu + \phi^{i}_{\lambda_1 \cdots \lambda_r}(y)\partial_i) \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$

(iii) Vertical-valued horizontal forms

$$\phi\colon \quad E \to \Lambda T^*B \underset{E}{\otimes} VE$$

$$\phi = \phi^{i}_{\lambda_1 \cdots \lambda_r}(y)\partial_i \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$

A vertical-valued horizontal 1-form is called a soldering form:

$$\sigma\colon \quad E \to T^*B \underset{E}{\otimes} VE$$
$$\sigma = \sigma^{i}_{\lambda}(y)\partial_i \otimes dx^{\lambda} \tag{18}$$

Tangent-valued 0-forms (i.e., vector fields) are known to form a sheaf of Lie algebras with respect to the commutation bracket. This algebra structure can be generalized to tangent-valued forms if we consider the Frölicher–Nijenhuis (FN) bracket:

$$\begin{aligned}[\phi, \sigma] = (&\phi^{\nu}_{\lambda_1 \cdots \lambda_r}\partial_\nu \sigma^{\mu}_{\lambda_{r+1} \cdots \lambda_{r+s}} - (-1)^{rs}\sigma^{\nu}_{\lambda_1 \cdots \lambda_s}\partial_\nu \phi^{\mu}_{\lambda_{s+1} \cdots \lambda_{r+s}} \\ &- r\phi^{\nu}_{\lambda_1 \cdots \lambda_{r-1}\nu}\partial_{\lambda_r}\sigma^{\nu}_{\lambda_{r+1} \cdots \lambda_{r+s}} + (-1)^{rs} s\sigma^{\mu}_{\lambda_1 \cdots \lambda_{s-1}\nu}\partial_{\lambda_s}\phi^{\nu}_{\lambda_{s+1} \cdots \lambda_{r+s}}) \\ &\times dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_{r+s}} \otimes \partial_\mu\end{aligned}$$

Given a tangent-valued form θ, we can introduce the Nijenhuis differential

$$d_\theta\colon \quad \sigma \mapsto d_\theta \sigma = [\theta, \sigma] \tag{19}$$

For instance, if $\theta = u$ is a vector field, we have the Lie derivative

$$d_u\sigma = L_u\sigma = (u^\nu \partial_\nu \sigma^\mu_{\lambda_1\cdots\lambda_s} - \sigma^\nu_{\lambda_1\cdots\lambda_s}\partial_\nu u^\mu + s\sigma^\mu_{\lambda_1\cdots\lambda_{s-1}\nu}\partial_{\lambda_s}u^\nu)\, dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_s}\otimes\partial_\mu \tag{20}$$

Note that the differential (19) can be applied to $\mathbb{R}$-valued forms σ.

3. JET MANIFOLDS

We here restrict ourselves to first-order and second-order jet manifolds.

Given a bundle E, the first-order jet manifold J^1E of E is defined to be made up of equivalence classes

$$w = j^1_x e, \qquad x \in B$$

of sections $e(x)$ of E so that $e(x)$ and $e'(x)$ belong to the same class $j^1_x e$ if and only if

$$e(x) = e'(x), \qquad e_*|_{T_xB} = e'_*|_{T_xB}$$

The jet manifold J^1E represents the total space of the bundles

$$E^1 = (J^1E, \pi_1, B), \qquad \pi_1\colon\ J^1E \ni j^1_x e \to x \in B$$

$$E^{01} = (J^1E, \pi_{01}, E), \qquad \pi_{01}\colon\ J^1E \ni j^1_x e \to e(x) \in E$$

Note that the structure of a smooth finite-dimensional manifold is induced on J^1E as on the bundle E^{01}.

Given bundle coordinates (x^λ, y^i) on E, the jet manifold J^1E is provided with adapted coordinates (2). Adapted coordinate transformations read

$$x'^\lambda = \Phi^\lambda_B(x^\mu) \tag{21a}$$

$$y'^i = \Phi^i(x^\mu, y^j) \tag{21b}$$

$$y'^i_\lambda = \left(\frac{\partial\Phi^i}{\partial y^j} y^j_\mu + \frac{\partial\Phi^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda} \tag{21c}$$

Note that the transformation law (21a) is independent of y^i and y^i_λ and that the transformation law (21b) does not involve y^i_λ. It follows that adapted coordinates on J^1E also play the role of bundle coordinates on J^1E regarded both as the bundle E^1 and as the bundle E^{01}.

Moreover, the second term in the transformation law (21c) indicates that E^{01} is an affine bundle. Namely, there is a canonical bundle monomorphism θ_1 of J^1E onto an affine subbundle of the bundle

$$T^*B \underset{E}{\otimes} TE$$

It is called the contact map and is given by the coordinate expression

$$\theta_1 = dx^\lambda \otimes d_\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) \tag{22}$$

The bundle E^{01} hence is the affine bundle modeled on the vector bundle

$$T^*B \underset{E}{\otimes} VE \to E \tag{23}$$

Let E and E' be bundles over B and

$$\Phi: \quad E \to E'$$

be some bundle morphism over a diffeomorphism Φ_B of B. Then, there exists a jet prolongation of a morphism Φ to the morphism

$$j^1\Phi: \quad J^1E \ni j^1_x e \to j^1_{\Phi_B(x)}(\Phi \circ e \circ \Phi_B^{-1}) \in J^1E'$$

For instance, each section e of E can be regarded as the bundle morphism of the bundle $B \to B$ into the bundle E over B. Hence, we have the jet prolongation of a section e to the section

$$(j^1e)(x) = j^1_x e$$

of the bundle E^1. In adapted coordinates, this prolongation reads

$$(x^\lambda, y^i, y^i_\lambda) \circ j^1 e = (x^\lambda, e^i(x), \partial_\lambda e^i(x))$$

In Section 6, we shall need the lift

$$\bar{u}: \quad J^1E \to TJ^1E$$

of a projectable vector field u on E to a vector field on J^1E. The coordinate expression of this lift is

$$\begin{gathered}\bar{u}(w) = u^\lambda\partial_\lambda + u^i\partial_i + (\partial_\lambda u^i + y^j_\lambda \partial_j u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i \\ x = \pi(y) = \pi_1(w), \qquad y = \pi_{01}(w)\end{gathered} \tag{24}$$

By analogy with 1-jet manifolds, higher-order jet manifolds can be introduced. Here we are concerned only with the second-order jet manifold J^2E of a bundle E. This is made up of equivalence classes $j^2_x e$ of sections $e(x)$ of E so that $e(x)$ and $e'(x)$ belong to the same class $j^2_x e$ if and only if

$$j^1_x e = j^1_x e', \qquad e_{**}|_{T_xTB} = e'_{**}|_{T_xTB}$$

By TTB, we here denote the tangent bundle over the tangent bundle TB.

The 2-jet manifold J^2E is endowed with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu} = y^i_{\mu\lambda})$, where

$$y^i_{\lambda\mu}(j^2_x e) = \partial_\mu \partial_\lambda y^i(e)$$

We can consider the repeated jet manifold

$$J^1J^1E \to B$$

provided with local coordinates

$$(x^\lambda, y^i, y^i_\lambda, y^i_{0\mu}, y^i_{\lambda\mu})$$

There are two bundle morphisms over J^1E:

$$\pi_{1(01)}: \quad J^1J^1E \to J^1E, \qquad y^i_\lambda \circ \pi_{1(01)} = y^i_\lambda$$

$$j^1\pi_{01}: \quad J^1J^1E \to J^1E, \qquad y^i_\lambda \circ j^1\pi_{01} = y^i_{0\lambda}$$

By recalling the affine structure of the bundle E^{01}, we find that their difference over J^1E yields the bundle morphism

$$j^1\pi_{01} - \pi_{1(01)} = \xi: \quad J^1J^1E \to T^*B \underset{E}{\otimes} VE$$

$$(x^\lambda, y^i, \dot{x}_\lambda \otimes \dot{y}^i) \circ \xi = (x^\lambda, y^i, y^i_{0\lambda} - y^i_\lambda)$$

The kernel of ξ is an affine subbundle $\hat{J}^2E \subset J^1J^1E$ over J^1E, which is characterized by the condition

$$y^i_{0\lambda} = y^i_\lambda$$

The adapted coordinates on $\hat{J}^2E$ are $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})$, where, in contrast with coordinates on J^2E,

$$y^i_{\lambda\mu} \neq y^i_{\mu\lambda}$$

Hence, there exist the following affine bundle monomorphisms over J^1E:

$$J^2E \to \hat{J}^2E \to J^1J^1E \tag{25}$$

and there is an affine splitting of $\hat{J}^2E$ over J^1E:

$$\hat{J}^2E = J^2E \underset{J^1E}{\oplus} \left(\overset{2}{\wedge} T^*B \underset{E}{\otimes} VE \right)$$

4. GENERAL CONNECTIONS

In general, a connection on a bundle E must determine the lift of a tangent vector to B at a point $x \in B$ to tangent vectors to E at each point $y \in E$ projected to x. In other words, a connection Γ on a bundle E can be viewed as a morphism

$$\Gamma: \quad E \underset{B}{\times} TB \to TE \tag{26}$$

One can introduce a connection in various equivalent ways. In the framework of jet formalism, we do it as follows.

Given a bundle E, a connection Γ on E is defined to be a global section (8) of the bundle E^{01}. Its coordinate expression is

$$(x^\lambda, y^i, y^i_\lambda) \circ \Gamma = (x^\lambda, y^i, \Gamma^i_\lambda(y))$$

Let Γ be a connection on a bundle E and Φ be a bundle isomorphism. Then,

$$\Gamma' = j^1\Phi \circ \Gamma \circ \Phi^{-1}$$

is a connection on E. In particular, if Φ is a bundle isomorphism over Id B, we have the coordinate expression

$$(x^\lambda, y'^i, y'^i_\lambda) \circ \Gamma' = (x^\lambda, y'^i, (\partial_\lambda \Phi^i + \Gamma^j_\lambda \partial_j \Phi^i) \circ \Phi^{-1})$$

By means of the contact map θ_1 of (22), a connection Γ can be viewed as a projectable tangent-valued horizontal form

$$\theta_1 \circ \Gamma: \quad E \to T^*B \underset{B}{\otimes} TE$$

We denote this form by the same symbol Γ. Its coordinate expression is

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda(y)\partial_i)$$

The form Γ determines the morphism (26):

$$\Gamma: \quad (y, \partial_\lambda) \to (\partial_\lambda + \Gamma^i_\lambda(y)\partial_i) \in T_yE$$

which yields the splitting of the exact sequence (13).

A connection Γ defines the bundle morphism

$$D: \quad J^1E \ni w \to w - \Gamma(\pi_{01}(w)) \in T^*B \underset{E}{\otimes} VE$$

of the affine bundle E^{01} into the vector bundle (23). We call this morphism a covariant differential. Its coordinate expression is

$$D = [y^i_\lambda - \Gamma^i_\lambda(y)]\, dx^\lambda \otimes \partial_i \tag{27}$$

To describe the totality of connections on a bundle E, one can use the following fact.

Proposition. Let Γ be a connection and σ be some soldering form (18) on E. Then, their affine sum

$$\Gamma' = \Gamma + \sigma: \quad E \to J^1E$$

$$\Gamma' = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda(y)\partial_i + \sigma^i_\lambda(y)\partial_i]$$

over E is a connection on E. Let Γ and Γ' be connections on E. Then, their affine difference over E is a soldering form.

The general approach to connections is suitable for formulating the classical concept of a principal connection. This is a connection A on a principal bundle P with a structure group G which obeys certain symmetries by the action of G on P and J^1P. Namely, a principal connection is a section of the bundle P^{01} which is a G-equivariant bundle morphism such that the following diagram is commutative for each morphism (10):

$$\begin{array}{ccc} P & \xrightarrow{A} & J^1P \\ {\scriptstyle r_g}\downarrow & & \downarrow{\scriptstyle j^1 r_g} \\ P & \xrightarrow{A} & J^1P \end{array}$$

Given an atlas Ψ^P and associated canonical coordinates (17) on P, we have

$$A = dx^\lambda \otimes (\partial_\lambda + A^m_\lambda(p)\partial_m)$$

$$A^m_\lambda(x^\lambda, p^m)\partial_m = (r_g)_*(A^m_\lambda(x^\lambda, 0)\partial_m) = A^m_\lambda(x^\lambda, 0) \text{ ad } g^{-1}(\partial_m)$$

In the case of a principal bundle P, the exact sequence (13) implies the exact sequence

$$0 \to V^G P \to T^G P \to TB \to 0$$

where

$$V^G P = VP/G, \qquad T^G P = TP/G$$

denote the quotients of VP and TP by the canonical action (10) of G on P. A principal connection A defines the splitting of this sequence.

Let E be a bundle associated with the principal bundle P. A principal connection A on P yields an associated principal connection on E. With respect to associated atlases Ψ^P of P and Ψ of E, a connection A on E takes the coordinate form

$$A^i_\lambda(y) = A^m_\lambda(x) I_m(y^i), \qquad A^m_\lambda(x) = A^m_\lambda(x^\lambda, 0) \tag{28}$$

By I_m, we here denote generators of the group G acting on a standard fiber F of E on the left.

5. GEOMETRIC THEORY OF CLASSICAL FIELDS

Let us examine matter fields ϕ identified with global sections of a vector bundle

$$(E, \pi, X, F, G)$$

over a world manifold X. This bundle is associated with a principal bundle P. We call E a matter bundle.

We further assume that X is an n-dimensional manifold endowed with a fiber metric

$$g: \quad X \to \overset{2}{\vee} TX$$

in the cotangent bundle T^*X and the dual fiber metric in TX, which we denote by the same symbol.

Given coordinates (v^i) on the standard fiber F of E, a bundle atlas Ψ yields linear bundle coordinates (x^λ, y^i), (9), on the vector bundle E. These coordinates are adapted to the canonical vertical splitting (16). Being endowed with this splitting, the vertical bundle VE is associated with the principal bundle P.

Let A be a principal connection on the principal bundle P. Given the associated principal connection A on E, the covariant differential (27) of field functions ϕ_κ reads

$$D\phi_\kappa = dx^\mu \otimes [\partial_\mu - A^m_\mu(x) I_m] \phi_\kappa(x)$$

where $A^m_\mu(x)$ are coefficients of the associated principal connection (28) on E.

Since principal connections on a principal bundle P with a structure group G are represented by G-equivariant sections of the jet bundle P^{01}, there is bijective correspondence between principal connections A on P and global sections A^C of the affine bundle

$$C = P^{01}/G = (J^1P/G \to P/G = X) \tag{29}$$

modeled on the vector bundle

$$\bar{C} = T^*X \otimes V^G P$$

We call C a connection bundle.

Sections of the bundle $V^G P$ are vertical vector fields on P invariant under the canonical action (10) of G on P on the right. This bundle is associated with the principal bundle P. Its standard fiber is the right Lie algebra $\tilde{\mathfrak{g}}$ of right-invariant vector fields on G. The structure group G acts on this standard fiber by the adjoint representation.

Given an atlas $\{z_\kappa\}$ of P, the bundle $V^G P$ is provided with associated bundle coordinates (x^μ, k^m) such that right-invariant vertical vector fields

$$u(p) = \dot{p}^m(z_\kappa(x) g) \partial_m = \dot{p}^m(z_\kappa(x)) \operatorname{ad} g^{-1}(\partial_m)$$

on P are represented by sections of the bundle $V^G P$:

$$u^C(x) = k^m(x) I_m = \dot{p}^m(z_\kappa(x)) I_m$$

where $\{I_m\}$ is a basis for the right Lie algebra $\tilde{\mathfrak{g}}$. The corresponding bundle coordinates on C are (x^μ, k^m_μ). A section A^C of the bundle C then has the coordinate expression

$$k^m_\mu \circ A^C = A^m_\mu(x)$$

In gauge theory, sections A^C are treated as gauge potentials.

Recalling the contact map (22), we may represent a section of the affine bundle (29) by the form

$$A^C: \quad X \to T^*X \otimes T^G P$$

$$A^C = dx^\mu \otimes [\partial_\mu - A^m_\mu(x) I_m]$$

The finite-dimensional configuration space of the matter fields ϕ is the jet manifold J^1E.

The configuration space of the gauge fields is the jet manifold J^1C. The affine bundle C^{01} admits the canonical splitting

$$J^1C = C_+ \underset{C}{\oplus} C_- = (J^2P/G) \underset{C}{\oplus} \left(\overset{2}{\wedge} T^*X \otimes V^G P \right) \tag{30}$$

where C_+ is the affine bundle modeled on the vector bundle

$$\bar{C}_+ = \overset{2}{\vee} T^*X \otimes V^G P \tag{31}$$

(Mangiarotti and Modugno, 1985). Local coordinates

$$(x^\mu, k^m_\mu, s^m_{\mu\lambda}, F^m_{\lambda\mu}) = (x^\mu, k^m_\mu, k^m_{\mu\lambda} + k^m_{\lambda\mu}, k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl} k^n_\lambda k^l_\mu) \tag{32}$$

on J^1C are adapted both to the submanifold C_+ and to the submanifold

$$C_- = C \underset{X}{\times} \left(\overset{2}{\wedge} T^*X \otimes V^G P \right)$$

Here, c^m_{nl} are the structure constants of the group G.

Remark. To get the splitting (30), one can use the monomorphisms (25) and the canonical isomorphism of $\hat{J}^2P/G$ to J^1C.

From the splitting (30), one obtains the fundamental form

$$\begin{aligned} F: \quad & J^1C \to \overset{2}{\wedge} T^*X \otimes V^G P \\ F = {} & \tfrac{1}{2}(k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl} k^n_\lambda k^l_\mu)\, dx^\lambda \wedge dx^\mu \otimes I_m \end{aligned} \tag{33}$$

If A is a principal connection on P, its curvature is given by

$$F_A = F \circ j^1 A^C$$

The gauge-invariant Lagrangians of gauge fields are known to be constituted only by the form (33), whereas the form

$$S: \quad J^1C \to C_+$$

is defined by gauge condition.

Given configuration spaces J^1E and J^1C of matter fields and gauge potentials, we can consider the first-order Lagrangian formalism.

6. LAGRANGIAN FORMALISM

A first-order Lagrangian is defined to be a morphism (1). We call $\mathscr{L}$ a Lagrangian density.

The following objects are usually associated with a Lagrangian.

1. The Legendre morphism. One calls the bundle (3) over E the Legendre bundle. This is provided with the standard coordinates:

$$\begin{aligned} \pi \circ \pi_\Pi: \quad & \Pi \to E \to X \\ (x^\lambda, y^i, p_i^\lambda) & \to (x^\lambda, y^i) \to (x^\lambda) \end{aligned} \tag{34}$$

The Legendre morphism (5) is defined to be the fiber derivative of L.

2. The Poincaré-Cartan form:

$$\Theta = \pi_i^\lambda \, dy^i \wedge \omega_\lambda - (\pi_i^\lambda y_\lambda^i - \mathscr{L})\omega \tag{35}$$

3. The Euler-Lagrange operator:

$$\mathscr{E}(L): \quad J^2E \to \overset{n}{\wedge} T^*X \wedge V^*E$$

$$\mathscr{E}(L) = (\partial_i \mathscr{L} - d_\lambda \pi_i^\lambda) \, dy^i \wedge \omega = \delta_i \mathscr{L} \, dy^i \wedge \omega$$

where by d_λ and δ_i we denote total derivatives and variation derivatives,

$$d_\lambda = \partial_\lambda + y_\lambda^i \partial_i + y_{\lambda\mu}^i \partial_i^\mu, \qquad \delta_i = \partial_i - d_\lambda \partial_i^\lambda$$

Given a Lagrangian L and the jet prolongation j^1e of a section e of E, the Euler-Lagrange equations read

$$(j^2e)^*[u \rfloor \mathscr{E}(L)] = 0 \tag{36}$$

$$(j^2e)^*[\delta_i \mathscr{L}] = \frac{\partial \mathscr{L}(e)}{\partial e^i} - \partial_\lambda \frac{\partial \mathscr{L}(e)}{\partial e^i_{,\lambda}} = 0 \tag{37}$$

for every vertical vector field u on E.

In field theory, a Lagrangian is usually required to be gauge invariant.

There are two main types of gauge transformations. These are atlas transformations and principal morphisms.

Here, we consider gauge transformations associated with internal symmetries which do not concern the tangent bundle TX over a base manifold X.

In field theory, atlas transformations are treated as transformations of reference frames. They do not act on sections $\phi(x)$ of the bundle E, but change their representation by field functions $\phi_\kappa(x)$.

Principal morphisms Φ_E of E are bundle morphisms (12) induced by principal isomorphisms (11) of the principal bundle P over the identity morphism of the base X. In contrast with atlas transformations, principal morphisms Φ_E alter sections of E.

The necessary condition of gauge invariance of a Lagrangian L consists in bringing L into zero by generators of infinitesimal principal morphisms. These generators are associated with certain vertical vector fields u_g on the bundle E. We call such a vector field a principal vertical vector field.

In order to define gauge generators acting on a Lagrangian, we can construct the lift (24) of corresponding principal vector fields:

$$\bar{u}_g = u_g^i\partial_i + (\partial_\lambda u_g^i + y_\lambda^j\partial_j u_g^i)\partial_i^\lambda$$

These generators act on $\mathbb{R}$-valued forms and tangent-valued forms on J^1E as Lie derivatives (20) given by the FN bracket:

$$L_{\bar{u}_g}\phi = d_{\bar{u}_g}\phi = [\bar{u}_g, \phi]$$

In particular, gauge generators act on a Lagrangian L by the rule

$$L_{\bar{u}_g}(L) = [\bar{u}_g, L] = (u_g^i\partial_i + (\partial_\lambda u_g^i + y_\lambda^j\partial_j u_g^i)\partial_i^\lambda)\mathscr{L}\omega \tag{38}$$

If $\mathscr{L}$ is gauge invariant, we have

$$L_{\bar{u}_g}\mathscr{L} = 0$$

for all principal vertical vector fields u_g. This equality makes sense of some conservation laws and provides us with certain conditions on the constitution of a gauge-invariant Lagrangian.

In the case of unbroken internal symmetries, the total Lagrangian L of the gauge theory is defined on the configuration space

$$J^1E \underset{X}{\times} J^1C$$

Let us provide this configuration space with the condensed coordinates

$$(x^\mu, q^A, q_\mu^A), \qquad q^A = (y^i, k_\mu^m)$$

and calculate the Lie derivative (38) of the total Lagrangian L:

$$L_{\bar{u}_g}L = [u_g^A\delta_A\mathscr{L} + d_\lambda(u_g^A\partial_A^\lambda\mathscr{L})]\omega = 0 \tag{39}$$

The local principal vertical field u_g on the bundle $E \times_X C$ takes the form

$$\begin{aligned} u_g &= [u_m^A(q^B)\alpha^m(x^\mu) + u_m^{A\lambda}(q^B)\partial_\lambda\alpha^m(x^\mu)]\partial_A \\ &= \alpha^m(x^\mu)I_{mj}^i y^j\partial_i + [\partial_\lambda\alpha^m(x^\mu) + c_{nl}^m k_\lambda^n\alpha^m(x^\mu)]\partial_m^\lambda \end{aligned}$$

where $\alpha^m(x^\mu)$ are arbitrary local functions on X. Substituting this expression into equality (39), one reproduces the familiar Noëther identities for a gauge-invariant Lagrangian:

$$u_m^A \delta_A \mathscr{L} + d_\mu (u_m^A \partial_A^\mu \mathscr{L}) = 0$$
$$u_m^{A\lambda} \delta_A \mathscr{L} + d_\mu (u_m^{A\lambda} \partial_A^\mu \mathscr{L}) + u_m^A \partial_A^\lambda \mathscr{L} = 0$$
$$u_m^{A\lambda} \partial_A^\mu \mathscr{L} + u_m^{A\mu} \partial_A^\lambda \mathscr{L} = 0$$

The total Lagrangian of the gauge theory is given by the sum

$$L = L_{(m)} + L_{(A)}$$

of the matter field Lagrangian $L_{(m)}$ and the Lagrangian $L_{(A)}$ of the gauge potentials.

We give an example of F-valued scalar matter fields. Let a^E be a G-invariant metric in F and Γ a connection on E. The familiar scalar field Lagrangian $L_{(m)}$ and the corresponding Euler-Lagrange operator read

$$L_{(m)} = \tfrac{1}{2}[g^{\mu\nu} a_{ij}^E (y_\mu^i - \Gamma_\mu^i(y))(y_\nu^j - \Gamma_\nu^j(y)) - m^2 a_{ij}^E y^i y^j] |g|^{1/2} \omega$$
$$\mathscr{E}(L) = -a_{ik}^E [m^2 y^i + g^{\mu\nu}(y_{\mu\nu}^i - y_\mu^j \partial_j \Gamma_\nu^i(y))] |g|^{1/2} \omega \otimes dy^k$$
$$\Gamma_\mu^i(y) = k_\mu^m I_{mj}{}^i y^j, \qquad g = \det g_{\mu\nu} \tag{40}$$

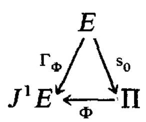
The conventional Yang-Mills Lagrangian $L_{(A)}$ of gauge potentials on the jet manifold J^1C provided with coordinates (32) is given by the expression

$$L_A = \frac{1}{4\varepsilon^2} a_{mn}^G g^{\lambda\mu} g^{\beta\nu} F_{\lambda\beta}^m F_{\mu\nu}^n |g|^{1/2} \omega \tag{41}$$

where a^G is the adjoint invariant metric in the Lie algebra $\tilde{\mathfrak{g}}$ and ε^2 is the coupling constant.

7. MULTIMOMENTUM HAMILTONIAN FORMALISM

Given a bundle E, let us consider the Legendre bundle Π of (3) and the commutative diagram



where s_0 is the global zero section of the bundle Π and Φ is a bundle morphism of Π to J^1E over E. Then,

$$\Gamma_\Phi = \Phi \circ s_0: \quad E \to J^1E$$

is a connection on E associated with Φ. We call Φ a momentum morphism. A momentum morphism can be canonically identified with a vector-valued horizontal 1-form

$$\theta_1 \circ \Phi: \quad \Pi \to T^*X \otimes TE$$

on Π, which we denote by the same symbol Φ. In standard coordinates (34) on Π, we have

$$\Phi(p) = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda(x^\lambda, y^i, p^\lambda_i)\partial_i)$$
$$\Gamma_\Phi(y) = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda(x^\lambda, y^i, 0)\partial_i)$$

We use p to denote elements of Π. By Π^1, we further denote the bundle $\Pi \to X$.

Given the Legendre bundle Π, there is a canonical inclusion

$$\Pi = \overset{n}{\wedge} T^*X \otimes TX \otimes V^*E \to \overset{n+1}{\wedge} T^*E \otimes TX$$

and so there is a canonical multimomentum Liouville form (4) on Π.

For each momentum morphism Φ, we then can define the associated multimomentum Hamiltonian form

$$\begin{aligned} H_\Phi &= \Phi \rfloor \theta: \quad \Pi \to \overset{n}{\wedge} T^*E \\ H_\Phi &= p^\lambda_i \, dy^i \wedge \omega_\lambda - p^\lambda_i \Phi^i_\lambda \omega \end{aligned} \tag{42}$$

Given the Legendre manifold Π, a multimomentum Hamiltonian form H on Π is defined by expression (6). Its transformation law (7) is derived from the following coordinate transformation rules:

$$\begin{gathered} x^\lambda \to x'^\lambda(x^\mu), \qquad y^i \to y'^i(x^\mu, y^j) \\ p'^\lambda_j = J \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial y^i}{\partial y'^j} p^\mu_i, \qquad dy'^j = \frac{\partial y'^j}{\partial x^\lambda} dx^\lambda + \frac{\partial y'^j}{\partial y^i} dy^i \\ \omega' = J^{-1} \omega, \qquad \omega'_\lambda = J^{-1} \frac{\partial x^\mu}{\partial x'^\lambda} \omega_\mu \end{gathered} \tag{43}$$

In particular, as it follows from the transformation law (7), the difference of two multimomentum Hamiltonian forms

$$H - H' = (\mathscr{H} - \mathscr{H}')\omega: \quad \Pi \to \overset{n}{\wedge} T^*X$$

is an exterior horizontal form on Π.

Given a multimomentum Hamiltonian form H on Π, one can define the associated momentum morphism $\hat{H}$ and the associated connection Γ_H on E:

$$(x^\lambda, y^i, y^i_\lambda) \circ \hat{H} = (x^\lambda, y^i, \partial^i_\lambda \mathscr{H})$$
$$\hat{H} = dx^\lambda \otimes (\partial_\lambda + \partial^i_\lambda \mathscr{H}(x^\lambda, y^i, p^\lambda_i)\partial_i)$$
$$\Gamma_H = dx^\lambda \otimes (\partial_\lambda + \partial^i_\lambda \mathscr{H}(x^\lambda, y^i, 0)\partial_i)$$

We can also construct the associated multimomentum Hamiltonian form

$$H_\Gamma = \Gamma \rfloor \theta$$
$$H_\Gamma = p_i^\lambda \, dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma^i_\lambda(y)\omega \tag{44}$$

for a connection Γ on E. Such a multimomentum Hamiltonian form has the feature that, if $\hat{H}_\Gamma$ is the associated momentum map and $H_{\hat{H}_\Gamma}$ is the associated multimomentum Hamiltonian form (42), then

$$H_{\hat{H}_\Gamma} = H_\Gamma$$

Conversely, if a multimomentum Hamiltonian form H satisfies condition

$$H_{\hat{H}} = H$$

this is the form (44) for some connection Γ on E.

Given a multimomentum Hamiltonian form H, there is a canonical splitting

$$H = p_i^\lambda \, dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma_H{}^i_\lambda(y)\omega - \tilde{\mathscr{H}}(p)\omega$$

where

$$\tilde{\mathscr{H}}_\omega = H_\Gamma - H: \quad \Pi \to \overset{n}{\wedge} T^*X$$

is an exterior horizontal n-form on Π. We call $\tilde{\mathscr{H}}$ a Hamiltonian density. Moreover, given a connection Γ on E and an associated multimomentum Hamiltonian form (44), each multimomentum Hamiltonian form H on Π can be written as

$$H = H_\Gamma - \tilde{\mathscr{H}}\omega \tag{45}$$

where $\tilde{\mathscr{H}}$ is some Hamiltonian density. In particular,

$$\Gamma_H = \Gamma + \sigma$$

where σ is some soldering form.

If H is a multimomentum Hamiltonian form (6) and r is a section of the bundle Π^1, the Hamiltonian equations for a section r take the form

$$r^*(c \rfloor dH) = 0 \tag{46}$$

for each vertical vector field c on the bundle Π^1. In standard coordinates, these equations are given by the expressions

$$\partial_\lambda r_i^\lambda = -\partial_i \mathscr{H} \tag{47a}$$

$$\partial_\lambda r^i = \partial^i_\lambda \mathscr{H} \tag{47b}$$

Remark. In view of relation (45), we can rewrite these equations in the form

$$D_\lambda r_i^\lambda = -\partial_i \tilde{\mathcal{H}}$$

$$D_\lambda r^i = \partial_i^\lambda \tilde{\mathcal{H}}$$

where D_λ are covariant derivatives corresponding to the connection $\tilde{\Gamma}$ on the bundle Π^1 which is induced by the connection Γ on E and the trivial connection on TX. In adapted coordinates $(x^\lambda, y^i, p_i^\mu, y_\lambda^i, p_{i\lambda}^\mu)$ on $J^1\Pi^1$, we have

$$\tilde{\Gamma}_\lambda^i(p) = \Gamma_\lambda^i(y)$$

$$\tilde{\Gamma}_{i\lambda}^\mu(p) = -\partial_i \Gamma_\lambda^j(y) p_j^\mu$$

To construct $\tilde{\Gamma}$, one can use the vertical tangent morphism

$$V\Gamma: \quad VE \to VJ^1E$$

and the canonical isomorphism

$$\beta: \quad VJ^1E \to J^1VE$$

Then, the connection $\tilde{\Gamma}$ is defined to be a connection on V^*E which is dual to the connection

$$\beta \circ V\Gamma: \quad VE \to TE \to VJ^1E \to J^1VE$$

on VE. If one uses covariant derivatives D_λ^T corresponding to a total connection on Π^1, the Hamiltonian equations read

$$D_\lambda^T r_i^\lambda - 2\Omega_{\mu\lambda}^\mu r_i^\lambda = \partial_i \tilde{\mathcal{H}}$$

$$D_\lambda^T r_i = \partial_i^\lambda \tilde{\mathcal{H}}$$

where Ω is the torsion of a connection on TX and T^*X.

Now we examine multimomentum Hamiltonian forms associated with Lagrangians.

Let L be a Lagrangian (1) on J^1E. For each momentum morphism Φ, we can define a multimomentum Hamiltonian form

$$H_{L\Phi} = H_\Phi + L \circ \Phi = p_i^\lambda \, dy^i \wedge \omega_\lambda - (p_i^\lambda \Phi_\lambda^i - \mathcal{L})\omega \tag{48}$$

By $Q = \hat{L}(J^1E)$, we denote the image of a Legendre morphism $\hat{L}$ associated with the Lagrangian L. We say that a momentum morphism Φ and a Hamiltonian form (48) are associated with the Lagrangian L if

$$\hat{L} \circ \Phi|_Q = \mathrm{Id}\, Q \tag{49}$$

and if the morphism $\hat{H}_{L\Phi}$ associated with the Hamiltonian form (48) coincides with Φ, that is,

$$[p_i^\lambda - \partial_i^\lambda \mathscr{L}(x^j, y^\mu, \Phi_\mu^k(p))]\partial_\nu^l \Phi_\lambda^i(p) = 0$$

On Q, this condition is the corollary of condition (49).

In particular, if $\hat{L}$ is a diffeomorphism, there exists the unique morphism

$$\Phi = \hat{L}^{-1}$$

associated to $\hat{L}$, and the multimomentum Hamiltonian form (48) is induced by a Poincaré–Cartan form (35):

$$H_{L\Phi} = H_L = (\hat{L}^{-1})^*\Theta$$
$$H_L(p) = p_i^\lambda \, dy^i \wedge \omega_\lambda - [p_i^\lambda y_\lambda^i(p) - \mathscr{L}(x^j, y^\mu, y_\mu^j(p))]$$
$$p_i^\lambda = \pi_i^\lambda = \partial_i^\lambda \mathscr{L}(x^j, y^\mu, y_\mu^j)$$

We further restrict our consideration to almost regular Lagrangians for which $\hat{L}^{-1}(p)$, $p \in Q$, is a connected submanifold of J^1E.

Proposition 1. Given an almost regular Lagrangian L, the Poincaré–Cartan form Θ defines uniquely the form

$$H_L\colon \quad Q \to \overset{n}{\wedge} T^*E$$

such that $\Theta = \hat{L}^* H_L$.

Outline of Proof. Since $\hat{L}(w) = p$, we have $\pi_i^\lambda = p_i^\lambda$ in expression (35). Hence, for any curve

$$\lambda\colon \quad (0, 1) \to J^1E$$

the formal derivative

$$\frac{d}{dt}\Theta(\lambda(t))$$

is equal to zero if λ lies in $\hat{L}^{-1}(p)$. Thus, $\Theta(w)$ does not depend on the choice of $w \in \hat{L}^{-1}(p)$.

Note that the set Q is analogous to a primary constraint manifold in the conventional Hamiltonian formalism.

It follows from Proposition 1 and condition (49) that, if the form (48) is associated with a Lagrangian L, then

$$H_{L\Phi}|_Q = H_L$$

A degenerate almost regular Lagrangian is exemplified by the zero Lagrangian $L = 0$. In this case, $Q = 0$ and multimomentum Hamiltonian forms (48) associated with this Lagrangian are exhausted by forms (44).

The following theorems relate the multimomentum Hamiltonian formalism and the Lagrangian formalism in the case of almost regular Lagrangians.

Proposition 2. Let r be a section of the bundle Π^1. If r takes its values in Q and satisfies Hamiltonian equations (46) for some $H_{L\Phi}$ associated with L, then

$$r = \hat{L}(j^1 e)$$

for some section e of E which satisfies the Euler–Lagrange equations (36).

Proposition 3. Let a section e of the bundle E be a solution of the Euler–Lagrange equations (36) and Φ be the momentum morphism associated with L such that

$$(\Phi \circ \hat{L}) j^1 e = j^1 e \tag{50}$$

Then,

$$r = \hat{L}(j^1 e)$$

satisfies the Hamiltonian equations (46) with $H = H_{L\Phi}$.

Outline of Proof. Equation (47a) can be easily derived from equations (37), and equation (47b) results from condition (50).

Remark. If the degeneracy rank of L is constant on Π, we can always construct a local morphism Φ satisfying the conditions of Proposition 3. Let $(V; x^\lambda, y^i, p_i^\lambda)$ be a chart of Q. In view of the constant degeneracy condition, we can select the maximal subset $\{\bar{y}_\lambda^i\}$ of coordinates $\{y_\lambda^i\}$ for which the equations

$$\bar{p}_i^\lambda = \frac{\partial \mathcal{L}}{\partial \bar{y}_\lambda^i}$$

can be resolved for $\bar{y}_\lambda^i$:

$$\bar{y}_\lambda^i = \bar{y}_\lambda^i(x^\mu, y^j, \bar{p}_j^\mu, \underline{y}_\mu^j) \tag{51}$$

where $\underline{y}_\lambda^i$ are the remaining coordinates. Since $\{\bar{y}_\lambda^i\}$ is the maximal subset of resolvable coordinates, after substituting (51) into the equation

$$\underline{p}_i^\lambda = \frac{\partial \mathcal{L}}{\partial \underline{y}_\lambda^i}$$

we obtain

$$\underline{p}_i^\lambda = \underline{p}_i^\lambda(x^\mu, y^j, \bar{p}_j^\mu)$$

Let $e=(x^\lambda, y^\mu(x))$ be a section of E. Then, the local morphism

$$V \ni (x^\lambda, y^i, p_i^\lambda) \to (x^\lambda, y^i, \bar{y}^i_\lambda(x^\mu, y^j, \bar{p}_j^\mu, \partial_\mu y^j(x)), \underline{y}^i_\lambda = \partial_\lambda \underline{y}^i)$$

satisfies the conditions of Proposition 3.

8. CONSERVATION LAW

To clarify the physical meaning of a multimomentum Hamiltonian form, we consider the following conservation law.

Let Γ be a connection on the bundle E, $e(x)$ be a section of E, and τ be a vector field on X. Given a Lagrangian L, coefficients of the form

$$\begin{aligned} \tau^\lambda T^\mu_\lambda(e)\omega_\mu &= -(j^1 e)^*(\tau_\Gamma \rfloor \Theta) \\ &= \tau^\lambda\{\pi_i^\mu[\partial_\lambda e^i - \Gamma^i_\lambda(e)] - \mathcal{L}(e)\delta^\mu_\lambda\}\omega_\mu \end{aligned}$$

$$\tau = \tau^\lambda(x)\partial_\lambda, \qquad \tau_\Gamma = \tau \rfloor \Gamma = \tau^\lambda(x)[\partial_\lambda + \Gamma^i_\lambda(y)\partial_i]$$

coincide with components of the canonical energy-momentum tensor of e projected onto τ. Let e be a solution of the Euler-Lagrange equations (36). If the Lie derivative

$$(j^2 e)^*[\bar{\tau}_\Gamma, L] = (j^2 e)^*[d_\lambda(\tau_\Gamma \rfloor \Theta)^\lambda + \tau^\lambda(\Gamma^i_\lambda - y^i_\lambda)\delta_i L]$$

vanishes, then the following energy-momentum conservation law holds:

$$\partial_\mu[\tau^\lambda T^\mu_\lambda(e)] = 0$$

In virtue of the splitting (45) and Proposition 2, one can bring this conservation law to the form

$$\tau^\lambda T^\mu_\lambda(r) = -r^*(\tau_\Gamma \rfloor H_L), \qquad r = \hat{L}(j^1 e)$$

$$\partial_\lambda\{[\tau^\mu r_i^\lambda(x) - \tau^\lambda r_i^\mu(x)]\partial^i_\mu \tilde{\mathcal{H}} + \tau^\lambda \tilde{\mathcal{H}}\} = 0$$

For instance, if $X = \mathbb{R}$, we have the familiar energy conservation law

$$\frac{d}{dt}\tilde{\mathcal{H}} = 0$$

9. CANONICAL TRANSFORMATIONS

By analogy with the familiar Hamiltonian formalism, let us consider canonical transformations of a multimomentum Hamiltonian system. They are defined to be transformations of bundle coordinates on the bundle Π^1 which keep the form of the Hamiltonian equations. We have treated the particular case (43) of canonical transformations. These are transformations of standard coordinates on Π which are bundle coordinates on Π^1 compatible with the fibration (34). As a consequence, we have obtained the canonical splitting (45).

Now, let us consider transformations of bundle coordinates on Π^1 of the following type:

$$x^\lambda \to x^\lambda, \qquad y^i \to y'^i(y^j, p_j^\mu), \qquad p_i^\lambda \to p_i'^\lambda(y^j, p_j^\mu) \tag{52}$$

These fail to preserve the fibration (34).

Let H be a multimomentum Hamiltonian form (6). We require that, for any local solution $(y^i(x), p_i^\lambda(x))$ of Hamiltonian equations associated with H, the local functions

$$y'^i(x) = y'^i(y^j(x), p_j^\mu(x))$$
$$p_i'^\lambda(x) = p_i'^\lambda(y^j(x), p_j^m(x))$$

be solutions of the Hamiltonian equations associated with the multimomentum Hamiltonian form

$$H' = p_i'^\lambda\, dy'^i \wedge \omega_\lambda - \mathcal{H}(x^\lambda, y^i(y'^j, p_j'^\mu), p_i^\lambda(y'^j, p_j'^\mu))\omega$$

Remark. This does not imply the equality

$$p_i^\lambda\, dy^i \wedge \omega_\lambda = p_i'^\lambda\, dy'^i \wedge \omega_\lambda$$

For instance, multimomentum Hamiltonian forms which differ from each other in an exact n-form result in the same Hamiltonian equations.

We can write

$$\partial_\lambda y'^i = \frac{\partial y'^i}{\partial p_k^\alpha}\partial_\lambda p_k^\alpha + \frac{\partial y'^i}{\partial y^k}\partial_\lambda y^k = \frac{\partial y'^i}{\partial p_k^\alpha}\partial_\lambda p_k^\alpha + \frac{\partial y'^i}{\partial y^k}\frac{\partial \mathcal{H}}{\partial p_k^\lambda}$$
$$= \frac{\partial y'^i}{\partial p_k^\alpha}\partial_\lambda p_k^\alpha + \frac{\partial y'^i}{\partial y^k}\left[\frac{\partial \mathcal{H}}{\partial y'^j}\frac{\partial y'^j}{\partial p_k^\lambda} + \frac{\partial \mathcal{H}}{\partial p_j'^\mu}\frac{\partial p_j'^\mu}{\partial p_k^\lambda}\right]$$

In contrast with derivatives $\partial_\lambda y^k$, derivatives $\partial_\lambda p_k^\alpha$ are not defined by Hamiltonian equations if $n > 1$. In order to get Hamiltonian equations for y'^i, one therefore must assume that, in expression (52), y'^i is independent of p_j^μ and that

$$\frac{\partial y'^k}{\partial y^k}\frac{\partial p_j'^\mu}{\partial p_k^\lambda} = \delta_\lambda^\mu \delta_j^i$$

This takes place only if the transformations (52) are reduced to the transformations (43). It follows that, if $n > 1$, the transformations (43) of standard coordinates on the Legendre bundle Π exhaust canonical transformations of a multimomentum Hamiltonian system.

By canonical maps, we call bundle morphisms of the bundle Π^1 which transform each solution of Hamiltonian equations into a solution. We restrict ourselves here to canonical maps over the identity morphism of X. In standard coordinates on the bundle Π^1, these canonical maps must take locally the form of the transformations (43) where $x'^{\mu} = x^{\mu}$. It follows that canonical maps are bundle morphisms of Π represented by bundle morphisms of the vertical contangent bundle V^*E which are induced by bundle morphisms of the bundle E.

Let us consider multimomentum Hamiltonian forms associated with an almost regular Lagrangian. By a gauge freedom transformation, we call a canonical map which transforms a multimomentum Hamiltonian form associated with L to a multimomentum Hamiltonian form associated with the same Lagrangian L. We have the following corollaries of Propositions 1-3.

(i) The induced multimomentum Hamiltonian form H_L on the image Q of the Legendre morphism is invariant under gauge freedom transformations. Moreover, in virtue of the splitting (45), the term H_Γ of (44) and the Hamiltonian density $\tilde{\mathcal{H}}$, each taken separately, must be invariant under gauge freedom transformations.

(ii) Gauge freedom transformations bring the Legendre images $\hat{L}(j^1 e)$ of local solutions e of the Euler-Lagrange equations into each other. Gauge freedom transformations thereby make sense of the transformations of physical equivalence.

10. MULTIMOMENTUM HAMILTONIANS OF GAUGE THEORY

In gauge theory, principal morphisms of a bundle E associated with some principal bundle P induce canonical maps of the Legendre manifold Π (we call them the gauge maps). To construct a multimomentum Hamiltonian formalism, let us require that the bundle E be endowed with an associated principal connection Γ and that a Hamiltonian density $\tilde{\mathcal{H}}$ in a multimomentum Hamiltonian form be invariant under gauge maps.

In the case of matter fields, we have a vector bundle E and a canonical vertical splitting

$$V^*E = E \times E^*$$

It follows that a Hamiltonian density can be constructed as a scalar function under linear gauge morphisms of bundles E and E^*. For instance, keeping the notations of Section 6, let us consider scalar matter fields. Its multimomentum Hamiltonian form reads

$$H_{(m)} = H_\Gamma - \tilde{\mathcal{H}}\omega = (p_i^\lambda \, dy^i - p_i^\lambda \Gamma^i_\lambda(y)\omega) - \tfrac{1}{2}(a^{ij}_E g_{\mu\nu} \tilde{p}^\mu_i \tilde{p}^\nu_j + m^2 a^E_{ij} y^i y^j)|g|^{1/2}\omega \tag{53}$$

where a_E is the fiber metric in E^* dual to a^E and

$$p = \tilde{p}|g|^{1/2}$$

This multimomentum Hamiltonian form is associated with the Lagrangian (40). The Hamiltonian density $\tilde{\mathcal{H}}$ in expression (53) is invariant under gauge maps. To construct a gauge-invariant multimomentum Hamiltonian form, however, one must add a multimomentum Hamiltonian form for principal connections and regard these connections as dynamic variables. In this case, gauge maps become gauge freedom transformations. Moreover, local gauge maps exhaust local gauge freedom transformations.

Given a bundle C, (29), of principal connections, we have the corresponding Legendre manifold

$$\begin{aligned}\Pi &= \overset{n}{\wedge} T^*X \otimes TX \underset{C}{\otimes} V^*C \\ &= \overset{n}{\wedge} T^*X \otimes TX \underset{C}{\otimes} [C \times (T^*X \otimes V^G P)]^* \\ &= \overset{n}{\wedge} T^*X \otimes \left[\left(\overset{2}{\vee} T^*X \otimes V^G P\right)^* \underset{C}{\oplus} \left(\overset{2}{\wedge} T^*X \otimes V^G P\right)^*\right] \\ &= \overset{n}{\wedge} T^*X \otimes \left[\bar{C}^*_+ \underset{C}{\oplus} C^*_-\right] \end{aligned} \tag{54}$$

where we use the vertical splitting

$$VC = C \times \bar{C} = C \times (T^*X \otimes V^G P)$$

We provide Π with standard coordinates

$$(x^\mu, k^m_\mu, p^{\mu\lambda}_m)$$

or

$$(x^\mu, k^m_\mu, p^{(\mu\lambda)}_m = \tfrac{1}{2}(p^{\mu\lambda}_m + p^{\lambda\mu}_m), p^{[\mu\lambda]}_m = \tfrac{1}{2}(p^{\mu\lambda}_m - p^{\lambda\mu}_m))$$

which are compatible with the splitting (54).

For the standard Yang-Mills Lagrangian $L_{(A)}$ of (41), the Legendre map reads

$$\begin{aligned} &\hat{L}_{(A)}: \quad J^1C \to Q = \overset{n}{\wedge} T^*X \otimes C^*_- \subset \Pi \\ &p^{[\mu\lambda]}_m = \varepsilon^{-2} a^G_{mn} g^{\lambda\alpha} g^{\mu\beta} F^n_{\alpha\beta} |g|^{1/2}, \qquad p^{(\mu\nu)} = 0 \end{aligned} \tag{55}$$

The multimomentum Hamiltonian form associated with the Lagrangian (41) reads

$$\begin{aligned} H_S &= p^{\mu\lambda}_m \, dk^m_\lambda \wedge \omega_\mu \\ &\quad - \tfrac{1}{2} p^{\mu\lambda}_m [S^m_{\mu\lambda}(x) - c^m_{nl} k^n_\lambda k^l_\mu]\omega - \frac{\varepsilon^2}{4} a^{mn}_G g_{\mu\nu} g_{\lambda\beta} \tilde{p}^{\mu\lambda}_m \tilde{p}^{\nu\beta}_n |g|^{1/2} \omega \end{aligned} \tag{56}$$

where $S(x)$ is an arbitrary section of the affine bundle C_+ and

$$\Gamma_C = \tfrac{1}{2}[S^m_{\mu\lambda}(x) - c^m_{nl} k^n_\lambda k^l_\mu]$$

is a connection on C. Thus, multimomentum Hamiltonian forms associated with the Lagrangian (41) differ from each other by a connection Γ_C.

Remark. Over $C \times J^1C$, there is canonical splitting of a connection on C:

$$\Gamma_C = \tfrac{1}{2}(s^m_{\mu\nu} - c^m_{nl} k^n_\lambda k^l_\mu) + \sigma^m_{\mu\nu}(x)$$

where $\sigma(x)$ is some section of the bundle (31).

For local sections $(k^m_\mu(x),\ p^{\mu\lambda}_m(x))$, the Hamiltonian equations associated with the multimomentum Hamiltonian form (56) read

$$\partial_\lambda p^{\mu\lambda}_n(x) = -\frac{\partial \mathcal{H}}{\partial k^n_\mu} = -\frac{1}{2} c^m_{ln} k^l_\lambda (p^{\lambda\mu}_m - p^{\mu\lambda}_m)$$

$$\partial_\lambda k^m_\mu(x) = \frac{\partial \mathcal{H}}{\partial p^{\mu\lambda}_m} = -\frac{1}{2}(S^m_{\mu\lambda}(x) - c^m_{nl} k^n_\lambda k^l_\mu) - \frac{\varepsilon^2}{2} a^{mn}_G g_{\mu\alpha} g_{\lambda\beta} \tilde{p}^{\alpha\beta}_n$$

On the image of the Legendre morphism (55), we have

$$H_S|_Q = H_L = p^{[\mu\lambda]}_m \, dk^m_\lambda \wedge \omega_\mu + \frac{1}{2} p^{[\mu\lambda]}_m c^m_{nl} k^n_\lambda k^l_\mu \omega$$

$$- \frac{\varepsilon^2}{4} a^{mn}_G g_{\mu\nu} g_{\lambda\beta} \tilde{p}^{[\mu\lambda]}_m p^{[\nu\beta]}_n |g|^{1/2} \omega$$

$$D_\lambda p^{[\mu\lambda]}(x) = 0$$

$$\partial_\lambda k^m_\mu + \partial_\mu k^m_\lambda = -S^m_{\mu\lambda}(x)$$

The last equation represents a gauge condition.

We thus can directly formulate gauge theory in the framework of the multimomentum Hamiltonian formalism.

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